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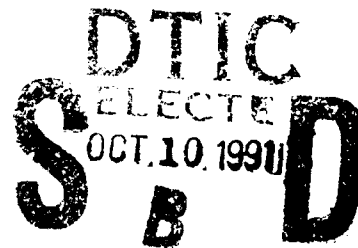
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PUSHOUTS AND GRAPH RECONSTRUCTION

BY A.D. PARKS
STRATEGIC SYSTEMS DEPARTMENT

JUNE 1991



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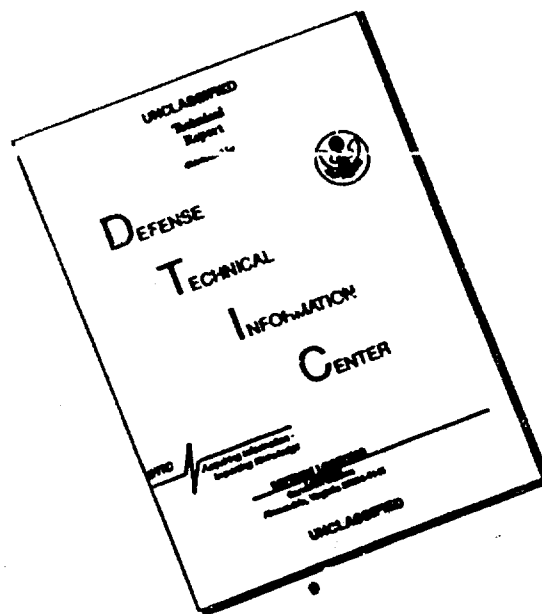
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FOREWORD

This report describes a certain "tangential" theoretical result obtained from work performed in the Strategic Systems (K) Department at the Naval Surface Warfare Center (NAVSWC) as part of an independent research grant entitled "A Category Theoretic Approach to Relational Database Schemes."

This report has been reviewed and approved by Ted Sims, Space Sciences Branch Head, and James L. Sloop, Space and Surface Systems Division Head.

Approved by:

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PRELIMINARIES

A graph $G = (V, E)$ is a finite nonempty set V of vertices and a set E of edges that are unordered pairs of elements of V . The order p of G is $|V|$, and the size q of G is $|E|$. The edge $e = \{u, v\} \in E$ joins the adjacent vertices u and v (denoted $u \sim v$), and is incident to both u and v .

The neighborhood $n(v)$ of a vertex v is the set of vertices adjacent to v . The number $d(v)$ of edges incident to v is the degree or valency of v . When $d(v) = 1$, then v is an end-vertex of G . The valency-sequence $vs(v)$ of a vertex v is the non-decreasing sequence of valencies of neighbors of v . The reduced valency-sequence $rvs(v)$ is obtained by subtracting one from each term of $vs(v)$. A $vs(u)$ for $u \in V(G - v)$ is v -excluded if $v \in n(u)$ in G .

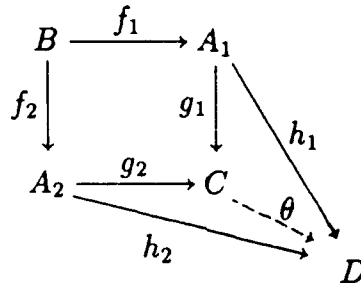
Ordinary graphs are those with order at least three. A vertex-deleted subgraph (or card) $G_v = G - v$ is the subgraph of G obtained by removing v and all edges incident to v . A null graph N_p has order p and size $q = 0$. A star graph $K_{1,s}$ is one that is isomorphic to the quotient graph obtained by identifying s edges at one common central vertex.

The category **Graph** has finite, simple, and undirected graphs for objects, and graph homomorphisms for morphisms. Here, $f : G \rightarrow H$ is a morphism when $f(u) \sim f(v)$ whenever $u \sim v$ (f is not necessarily onto).

In any category, a solution of the diagram

$$A_2 \xleftarrow{f_2} B \xrightarrow{f_1} A_1 \quad (\Delta)$$

is an object C and morphisms g_1, g_2 such that $g_1 f_1 = g_2 f_2$. A pushout of (Δ) is a solution (C, g_1, g_2) with the additional property that for any solution (D, h_1, h_2) , there exists a unique morphism $\theta : C \rightarrow D$ making the following diagram commute [1,p.33]:



If (C, g_1, g_2) and (D, h_1, h_2) are pushouts of (Δ) , then θ is an isomorphism [1,p.16]. This is the universal property of pushouts.

If O_1 and O_2 are objects in some category, then their coproduct is an object $O_1 \amalg O_2$ together with morphisms $h_k : O_k \rightarrow O_1 \amalg O_2$, $k = 1, 2$, called injections, with the property

that for any object X and any morphisms $m_k : O_k \rightarrow X$, $k = 1, 2$, there is a unique morphism $\psi : O_1 \amalg O_2 \rightarrow X$ such that $m_k = \psi h_k$, $k = 1, 2$ [3,p.58]. A coproduct in **Graph** is just the disjoint union of graphs together with the inclusion maps [2,p.128]. Specifically, if G and H are graphs, then $G \amalg H$ has as vertex set the disjoint union $V(G) \vee V(H)$; edge set the disjoint union $E(G) \vee E(H)$; and injections (imbeddings) $V(G) \hookrightarrow V(G) \vee V(H)$ and $V(H) \hookrightarrow V(G) \vee V(H)$.

An *automorphism* of a graph G is an isomorphism of G onto itself, i.e., a permutation of V which preserves adjacency. Collectively, the automorphisms of G (under composition) form the *automorphism* group $Aut(G)$ of G . Subsets V' and V'' of V are α -*equivalent* whenever there is an $\alpha \in Aut(G)$ mapping V' into V'' . If for each pair, (u_1, \dots, u_k) and (v_1, \dots, v_k) , of k -tuples of vertices of G there is an automorphism α of G such that $\alpha(u_i) = v_i$, $i = 1, \dots, k$, then the graph G is k -*transitive*.

A *hypomorphism* of a graph G onto a graph H is a bijection $\sigma : V(G) \rightarrow V(H)$ such that for every $v \in V(G)$, G_v is isomorphic to $(\simeq) H_{\sigma(v)}$. Graphs G and H are *hypomorphic* whenever a hypomorphism of G onto H exists. If G and H are hypomorphic ordinary graphs, then they are both connected or disconnected; they have the same number of vertices and the same number of edges; they have the same degree sequence; and corresponding vertices ($v \mapsto \sigma(v)$) have the same valency-sequence [4].

A graph G is *reconstructible* if every graph hypomorphic to G is isomorphic to G . If $z \in V(G)$, then a graph H is a z -*reconstruction* of G when $V(G) = V(H)$, $G - z = H - z$, and G is hypomorphic to H .

CARD DIAGRAMS AND CONJUGATE σ -PAIRS

We consider only ordinary graphs. Let G be such a graph with vertex v . Let $K = K_{1,d(v)}$ be the subgraph of G induced by the edges incident to v ; $N = N_{d(v)} = V(G_v) \cap V(K)$; and define the G_v *card diagram*

$$G_v \xhookrightarrow{i} N \xhookrightarrow{j} K, \quad (*)$$

where the inclusions i and j are clearly homomorphisms ($E(N) = \emptyset$).

The proof of the following Lemma is obvious.

Lemma 1. *If $\sigma : G \rightarrow H$ is a hypomorphism, then for each $(*)$ there is an induced $H_{\sigma(v)}$ card diagram*

$$H_{\sigma(v)} \xhookrightarrow{i'} N' \xhookrightarrow{j'} K', \quad (\#)$$

where $N' = N_{d(\sigma(v))}$, $K' = K_{1,d(\sigma(v))}$, and i', j' are inclusions.

For a hypomorphism $\sigma : G \rightarrow H$, the set $P(\sigma)$ contains p σ -pairs $\{i, i'\}$ where i and i' are corresponding inclusions appearing, respectively, in the G_v and $H_{\sigma(v)}$ card diagrams. If there are isomorphisms ϕ and ψ such that diagram $(@)$ commutes, then call i and i'

σ -conjugate.

$$\begin{array}{ccc}
 G_v & \xleftarrow{i} & N \\
 \phi \downarrow & & \downarrow \psi \\
 H_{\sigma(v)} & \xleftarrow{i'} & N'
 \end{array} \quad (Q)$$

Lemma 2. The inclusions i and i' are σ -conjugate if, and only if, there is an isomorphism $\phi : G_v \rightarrow H_{\sigma(v)}$ such that $\text{im}(\phi i) = \text{im}(i')$.

PROOF: (\Leftarrow) Let $\psi = (i')^{-1} \phi i$. (\Rightarrow) Obvious. ■

The now obvious proofs of the two following corollaries are omitted.

Corollary 3. Let σ, i, i' , be as in Lemma 2. Suppose $\phi : G_v \rightarrow H_{\sigma(v)}$ is an isomorphism. If there is an automorphism α of G such that $\text{im}(i)$ and $\text{im}(\phi^{-1} i')$ are α -equivalent, then i and i' are σ -conjugate.

Corollary 4. With the same hypothesis as Lemma 2, if i and i' are onto, then they are σ -conjugate.

The next Lemma also has an easy proof.

Lemma 5. Given any isomorphism $\psi : N \rightarrow N'$, there is an isomorphism $\gamma : K \rightarrow K'$ such that $\gamma j = j' \psi$.

MAIN RESULTS

Continuing with the same notation as above, let

$$\pi : K \coprod G_v \rightarrow G \quad \text{and} \quad \pi' : K' \coprod H_{\sigma(v)} \rightarrow H$$

be the projective homomorphisms which map the vertices and edges of the coproduct onto G and H . For example, the restriction of π to K is just an inclusion map. Next consider the diagram

$$\begin{array}{ccc}
 N & \xrightarrow{j} & K \\
 i \downarrow & & \downarrow h_1 \\
 G_v & \xrightarrow{h_2} & K \coprod G_v \\
 & \searrow \lambda_2 & \swarrow \lambda_1 \\
 & & G
 \end{array} \quad (1)$$

where the homomorphisms $\lambda_1 = \pi h_1$ and $\lambda_2 = \pi h_2$. Similarly, the π' morphism induces a corresponding diagram with λ'_1 and λ'_2 .

Lemma 6. $(G, \lambda_1, \lambda_2)$ is a pushout of $(*)$ in **Graph**.

PROOF: It is easy to show that $(G, \lambda_1, \lambda_2)$ is a solution to $(*)$. To see that $(G, \lambda_1, \lambda_2)$ is a pushout, consider the diagram

$$\begin{array}{ccc}
 N & \xrightarrow{j} & K \\
 i \downarrow & & \downarrow \lambda_1 \\
 G_v & \xrightarrow{\lambda_2} & G \\
 & \searrow \delta_2 & \nearrow \theta \\
 & & H
 \end{array}
 \quad (2)$$

Given that (H, δ_1, δ_2) is a solution, one can define the desired morphism

$$\theta = (\delta_1 \amalg \delta_2) \circ \pi^{-1}$$

where $\delta_1 \amalg \delta_2 : K \amalg G_v \rightarrow H$ exists because $K \amalg G_v$ is a coproduct. ■

Theorem 7. An ordinary graph G is reconstructible if, and only if, for each hypomorph H of G there are homomorphisms δ_1 and δ_2 such that (H, δ_1, δ_2) is a pushout of $(*)$ for some card of G .

PROOF: If G is reconstructible, then we can choose θ in diagram (2) to be an isomorphism and define

$$\delta_1 = \theta \lambda_1 \quad \text{and} \quad \delta_2 = \theta \lambda_2.$$

Since $(G, \lambda_1, \lambda_2)$ is a pushout of $(*)$, we see (H, δ_1, δ_2) is a pushout of $(*)$. The converse follows from the universal property of pushouts of $(*)$, i.e., any two are isomorphic. ■

Theorem 8. Let G be an ordinary graph. If for each hypomorphism σ , $P(\sigma)$ contains a conjugate σ -pair, then G is reconstructible.

PROOF: If i, i' are σ -conjugate, then there exists isomorphisms ϕ and ψ such that

$$\begin{array}{ccc}
 G_v & \xleftarrow{i} & N \\
 \phi \downarrow & & \downarrow \psi \\
 H_{\sigma(v)} & \xleftarrow{i'} & N'
 \end{array}
 \quad (3)$$

is commutative. Also, it follows from Lemma 5 that there exists an isomorphism γ which extends the diagram (3) to the commutative diagram:

$$\begin{array}{ccccc}
 G_v & \xleftarrow{i} & N & \xrightarrow{j} & K \\
 \phi \downarrow & & \downarrow \psi & & \downarrow \gamma \\
 H_{\sigma(v)} & \xleftarrow{i'} & N' & \xrightarrow{j'} & K'
 \end{array}
 \quad (4)$$

It follows from a proof analogous to that of Lemma 6, that for the inclusions $\delta_1 : K' \hookrightarrow H$, $\delta_2 : H_{\sigma(v)} \hookrightarrow H$, the triple (H, δ_1, δ_2) is a pushout of the bottom row of diagram (4). With this and the isomorphisms in diagram (4), we conclude that (H, δ_1, δ_2) is a pushout of $(*)$. But since H is an arbitrary hypomorph of G , we apply Theorem 7 and conclude that G is reconstructible. ■

APPLICATIONS

Assuming all graphs are ordinary, applications of Theorem 8, and the lemmas and corollaries of the second section, yield the following:

Theorem 9. *Let G be a graph of order p . If there is a $v \in V(G)$ such that $d_G(v) = p - 1$, then G is reconstructible.*

PROOF: Diagram (@) commutes for each hypomorph H , since $d_H(\sigma(v)) = p - 1$ and i, i' are onto. ■

Theorem 10. *A graph G which has a $d_G(v)$ -transitive G_v card is reconstructible.*

PROOF: Observe that $\text{im}(i)$ and $\text{im}(\phi^{-1}i')$ are α -equivalent in diagram (@) for each hypomorph. ■

Theorem 11. *A graph G is reconstructible if every hypomorph of G is a z -reconstruction of G .*

PROOF: For each hypomorph, ϕ and ψ are identities and diagram (@) commutes. ■

The following two theorems are equivalent to the "good neighbor" theorems found in [4, pp. 216-217].

Theorem 12. *A graph is reconstructible if it has a vertex with a unique reduced valency sequence.*

PROOF: A unique $\mathbf{rvs}_G(v)$ implies the corresponding set

$$\{d_{G_v}(w) \mid w \in \mathbf{n}_G(v)\}$$

is unique in G_v . Since ϕ preserves this uniqueness, diagram (@) commutes for each hypomorph. ■

Theorem 13. *A graph is reconstructible if it has a vertex v each neighbor of which has a unique, possibly v -excluded, valency sequence.*

PROOF: Observe that

$$\mathbf{vs}_{G_v}(w), \quad \text{where } w \in \mathbf{n}_G(v),$$

is preserved by ϕ and the diagram (@) commutes for each hypomorph. ■

REMARKS

The Reconstruction Conjecture, which asserts that every ordinary graph can be uniquely reconstructed from its deck of vertex deleted subgraphs, is generally regarded as one of the foremost unsolved problems in graph theory. Here we have cast the reconstruction problem in a category theoretic framework and demonstrated the ease with which certain reconstruction results can be obtained when viewed from this perspective. We note that while seeming of little practical value, it is possible that such results may prove useful in areas such as distributed information synthesis.

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